

Math 221-Midterm 1 Practice exam 2

1. (20 points) Calculate the derivatives of the following functions by using differentiation rules.

(a) $y = x^5 - \frac{5}{\sqrt[5]{x}} + \frac{1}{3x^4} = x^5 - 5x^{-\frac{1}{5}} + \frac{1}{3}x^{-4}$

$$\frac{dy}{dx} = 5x^4 + x^{-\frac{6}{5}} - \frac{4}{3}x^{-5}$$

(b) $y = \frac{\sin x}{x^3 + 1}$

$$\frac{dy}{dx} = \frac{\cos x \cdot (x^3 + 1) - \sin x \cdot 3x^2}{(x^3 + 1)^2}$$

(c) $y = (x^3 + x + 5)^{12}$

$$\frac{dy}{dx} = 12(x^3 + x + 5)^{11} \cdot (3x^2 + 1)$$

(d) $y = \frac{x}{\sqrt{1 + \cos^2 x}}$

$$\frac{dy}{dx} = \frac{\sqrt{1 + \cos^2 x} - x \frac{d}{dx} \sqrt{1 + \cos^2 x}}{(1 + \cos^2 x)}$$

$$= \frac{\sqrt{1 + \cos^2 x} + x \sin x \cos x \cdot (1 + \cos^2 x)^{-\frac{1}{2}}}{1 + \cos^2 x}$$

$$\begin{aligned} \frac{d}{dx} \sqrt{1 + \cos^2 x} &= \frac{1}{2} (1 + \cos^2 x)^{-\frac{1}{2}} \cdot 2 \cos x \cdot (-\sin x) \\ &= -\sin x \cos x \cdot (1 + \cos^2 x)^{-\frac{1}{2}} \end{aligned}$$

2. (9 points) Use the **definition of the derivative as a limit** to find the equation of the tangent line to the curve $y = \sqrt{3x^2 + 1}$ at $x = 1$.

The derivative of y at $x=1$

$$\begin{aligned} \frac{dy}{dx} \Big|_{x=1} &= \lim_{x \rightarrow 1} \frac{\sqrt{3x^2+1} - \sqrt{3+1}}{x-1} = \lim_{x \rightarrow 1} \frac{\sqrt{3x^2+1} - 2}{x-1} \\ &= \lim_{x \rightarrow 1} \frac{(\sqrt{3x^2+1} - 2)(\sqrt{3x^2+1} + 2)}{(x-1)(\sqrt{3x^2+1} + 2)} \\ &= \lim_{x \rightarrow 1} \frac{3x^2 + 1 - 4}{(x-1)(\sqrt{3x^2+1} + 2)} \\ &= \lim_{x \rightarrow 1} \frac{3x^2 - 3}{(x-1)(\sqrt{3x^2+1} + 2)} \\ &= \lim_{x \rightarrow 1} \frac{3(x+1)(x-1)}{(x-1)(\sqrt{3x^2+1} + 2)} \\ &= \frac{3 \cdot 2}{\sqrt{4} + 2} = \frac{3}{2} \end{aligned}$$

At $x=1$, $y=2$

$\Rightarrow$ equation of tangent at $x=1$: $y = \frac{3}{2}(x-1) + 2$

3. (25 points) Compute the following limits or show that it does not exist. If you think the limit does not exist, explain why, and specify if the limit is ∞ or $-\infty$.

(a) $\lim_{x \rightarrow 3} \frac{x^2 - 2x - 3}{x^3 - 9x}$

$$\begin{aligned} &= \lim_{x \rightarrow 3} \frac{(x+1)(x-3)}{x(x^2-9)} = \lim_{x \rightarrow 3} \frac{(x+1)(x-3)}{x(x+3)(x-3)} = \lim_{x \rightarrow 3} \frac{x+1}{x(x+3)} = \frac{4}{3 \cdot 6} = \frac{2}{9} \end{aligned}$$

$$(b) \lim_{x \rightarrow 2} \frac{\frac{1}{x^2} - \frac{1}{4}}{x - 2}$$

$$= \lim_{x \rightarrow 2} \frac{\frac{4 - x^2}{4x^2}}{x - 2} = \lim_{x \rightarrow 2} \frac{(2-x)(2+x)}{\frac{4x^2}{x-2}} = \lim_{x \rightarrow 2} -\frac{(x+2)}{4x^2} = -\frac{4}{4 \cdot 2^2} = -\frac{1}{4}$$

$$(c) \lim_{x \rightarrow 0} \frac{\tan 3x}{\sin 7x} \quad (\text{Hint: } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1.)$$

$$= \lim_{x \rightarrow 0} \frac{\sin 3x / \cos 3x}{\sin 7x} = \lim_{x \rightarrow 0} \frac{1}{\cos 3x} \cdot \frac{\sin 3x}{\sin 7x} = \lim_{x \rightarrow 0} \frac{1}{\cos 3x} \cdot \frac{\frac{\sin 3x}{3x} \cdot 3x}{\frac{\sin 7x}{7x} \cdot 7x}$$

$$= \frac{3}{7}$$

$$(d) \lim_{x \rightarrow 0} x^3 \cos\left(\frac{1}{x^2}\right)$$

$$-1 \leq \cos\left(\frac{1}{x^2}\right) \leq 1$$

$$0 \leq \left| x^3 \cos\left(\frac{1}{x^2}\right) \right| \leq |x^3|$$

$$-|x^3| \leq x^3 \cos\left(\frac{1}{x^2}\right) \leq |x^3|$$

$$\lim_{x \rightarrow 0} |x^3| = \lim_{x \rightarrow 0} -|x^3| = 0 \Rightarrow \lim_{x \rightarrow 0} x^3 \cos\left(\frac{1}{x^2}\right) = 0 \text{ by Sandwich theorem}$$

(e) $\lim_{x \rightarrow \infty} \frac{x^2 + 4x}{2x - 3}$

$= \lim_{x \rightarrow \infty} \frac{x+4}{2-\frac{3}{x}} = \infty$

4. (10 points) For each of the problems, circle the (unique) correct answer.

(a) One of the following functions is continuous but not differentiable at $x = 0$. Which one?

A. $f(x) = x|x|$ diff at $x=0$

B. $f(x) = \sin(\frac{1}{x})$ not continuous at $x=0$

☒ C. $f(x) = x^{1/3}$

D. $f(x) = \tan x$ diff at $x=0$

(b) $\lim_{x \rightarrow 4} \frac{x}{x^2 - 5x + 4}$ is

A. ∞

B. $-\infty$

C. 4.

☒ D. None of the above.

$\lim_{x \rightarrow 4} \frac{x}{(x-1)(x-4)} = \infty$, $\lim_{x \rightarrow 4} \frac{x}{(x-1)(x-4)} = -\infty$

(c) True or false? We know $\lim_{x \rightarrow 0} \frac{|x|}{x}$ does not exist because the bottom becomes 0 when $x = 0$.

A. True

☒ B. False

it DNE since $\lim_{x \rightarrow 0^+} \frac{|x|}{x} = 1$ but $\lim_{x \rightarrow 0^-} \frac{|x|}{x} = -1$

(d) True or false? If $\lim_{x \rightarrow 0} f(x) = 2$, then $f(0) = 2$.

A. True

☒ B. False

(e) True or false? There are functions whose graph crosses its asymptotes.

☒ A. True

B. False

5. (18 points) Do the following problems about finding asymptotes.

(a) (8 points) Find all the horizontal asymptotes of $f(x) = \frac{x}{\sqrt{x^2+4}}$. Show your work.

$$\lim_{x \rightarrow \infty} \frac{x}{\sqrt{x^2+4}} = \lim_{x \rightarrow \infty} \frac{x/x}{\sqrt{x^2+4}/x} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{\frac{x^2+4}{x^2}}} = \lim_{x \rightarrow \infty} \frac{1}{\sqrt{1+\frac{4}{x^2}}} = 1$$

$$\lim_{x \rightarrow -\infty} \frac{x}{\sqrt{x^2+4}} = \lim_{x \rightarrow -\infty} \frac{x/x}{\sqrt{x^2+4}/x} = \lim_{x \rightarrow -\infty} \frac{1}{\sqrt{x^2+4}/\sqrt{x^2}} = \lim_{x \rightarrow -\infty} -\frac{1}{\sqrt{\frac{x^2+4}{x^2}}} = -1$$

$\Rightarrow f(x)$ has two H.A. $y=1$ and $y=-1$

(b) (10 points) Find all the asymptotes of $f(x) = \frac{x^2+2x}{x+1}$. Show your work.

Slanted asymptotes: $\lim_{x \rightarrow \infty} \frac{f(x)}{x} = \lim_{x \rightarrow \infty} \frac{x^2+2x}{x^2+x} = \lim_{x \rightarrow \infty} \frac{1+\frac{2}{x}}{1+\frac{1}{x}} = 1$

$$\lim_{x \rightarrow \infty} [f(x) - x] = \lim_{x \rightarrow \infty} \left(\frac{x^2+2x}{x+1} - \frac{x^2+x}{x+1} \right) = \lim_{x \rightarrow \infty} \frac{x}{x+1} = 1$$

$\Rightarrow f(x)$ is asymptotic to $y=x+1$ as $x \rightarrow \infty$ and $x \rightarrow -\infty$

V.A: $f(x)$ is not continuous when $x=-1$

$$\lim_{x \rightarrow -1} \frac{x^2+2x}{x+1} = -\infty \Rightarrow f(x) \text{ has one V.A } x=-1$$

6. (18 points) Consider the function $f(x)$ defined by $f(x) = \begin{cases} (x+a)^2 & \text{for } x \leq 0 \\ 2x+b & \text{for } x > 0 \end{cases}$ and answer the following questions. Here a and b are parameters.

(a) State, in terms of limits, what it means for $f(x)$ to be continuous at $x = 0$.

$$\begin{aligned} & \Updownarrow \\ & \lim_{x \rightarrow 0} f(x) = f(0) \\ & \Updownarrow \\ & \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^-} f(x) = f(0) \end{aligned}$$

(b) If $f(x)$ is continuous at $x = 0$, we should get a relation between a and b . What is that relation?

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} 2x + b = b$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (x+a)^2 = a^2$$

$$f(0) = (0+a)^2 = a^2$$

$$\text{So } f \text{ is continuous at } x=0 \Leftrightarrow a^2 = b$$

(c) For which values of a and b is $f(x)$ differentiable at $x = 0$? Justify your answer.

$$\text{"Right derivative"} \quad \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} = \frac{d}{dx} (2x+b) \Big|_{x=0} = 2$$

$$\text{"Left derivative"} \quad \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \frac{d}{dx} (x+a)^2 \Big|_{x=0} = 2(x+a) \Big|_{x=0} = 2a$$

$$\text{So } \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} \text{ exists } \Leftrightarrow 2a = 2 \Leftrightarrow a = 1 \text{ and } b = a^2 = 1$$

$$\text{i.e. } f(x) \text{ is diff at } x=0 \Leftrightarrow a=1, b=1$$